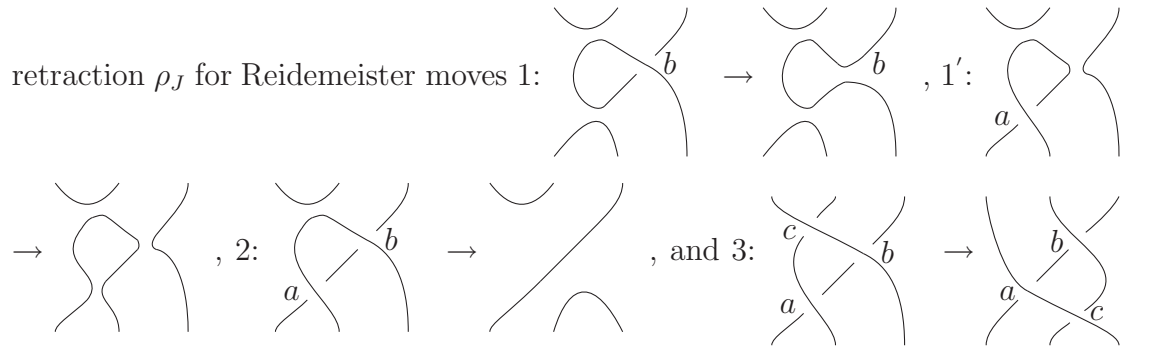


# SOME CHAIN MAPS ON KHOVANOV COMPLEXES AND REIDEMEISTER MOVES

NOBORU ITO

ABSTRACT. We introduce some chain maps between Khovanov complexes. Each of the chain maps commutes with a chain homotopy map and a retraction maps which obtain a Reidemeister invariance of Khovanov homology.

## 1. INTRODUCTION.

Let  $\mathcal{C}_3 = \mathcal{C} \left( \begin{array}{c} \text{diagram with strands } a, b, c \end{array} \right)$  be a Khovanov complex of a link diagram and  $\mathcal{C}_2$   
 $= \mathcal{C} \left( \begin{array}{c} \text{diagram with strands } a, b \end{array} \right)$ ,  $\mathcal{C}_1 = \mathcal{C} \left( \begin{array}{c} \text{diagram with strand } b \end{array} \right)$ , and  $\mathcal{C}_{1'} = \mathcal{C} \left( \begin{array}{c} \text{diagram with strand } a \end{array} \right)$  be its  
subcomplexes. There exist chain homotopy maps  $h_J$  ( $J = 1, 1', 2, 3$ ) relating the  
identity maps  $\mathcal{C}_J \rightarrow \mathcal{C}_J$  with compositions in  $\circ \rho_J$  of an inclusion maps in and a  
retraction  $\rho_J$  for Reidemeister moves 1:  


(Section 4, Appendix A, B). This paper will be show that the natural chain maps  $\pi_J: \mathcal{C}_J \rightarrow \mathcal{C}_{J-1}$  ( $J = 2, 3$ ) satisfy the relations  $h_{J-1} \circ \pi_J = \pi_J \circ h_J$  (Theorem 1) and similar relations for  $\rho_J$  (Theorem 2).

In section 1 chain maps  $\pi_J$  are defined. In section 2 relations of  $h_J$ ,  $\rho_J$ , and  $\pi_J$  are given. In section 3 a map  $\tilde{\pi}_2$  similar to  $\pi_2$  is introduced. In section 4 we obtain the proof of the right twisted first Reidemeister invariance of Khovanov homology for a general differential. Appendix contains the definitions  $h_J$  and  $\rho_J$  provided by [1, 2]. All notations in this paper and the definition of the differential  $\delta_{s,t}$  follows [2].

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## 2. THE CHAIN MAP $\pi_J$ .

$\pi_3 : \mathcal{C}_3 \rightarrow \mathcal{C}_2$  is defined by

$$(1) \quad \begin{array}{c} \begin{array}{ccc} \begin{array}{c} \text{Diagram 1: A crossing with strands } a, b, c. A blue dot is on strand } c \text{ above the crossing.} \end{array} & \mapsto & \begin{array}{c} \text{Diagram 2: A crossing with strands } a, b. \end{array} \\ \begin{array}{c} \text{Diagram 3: A crossing with strands } a, b, c. A red dot is on strand } c \text{ below the crossing.} \end{array} & \mapsto & 0. \end{array}$$

$\pi_2 : \mathcal{C}_2 \rightarrow \mathcal{C}_1$  is defined by

$$(2) \quad \begin{array}{c} \begin{array}{ccc} \begin{array}{c} \text{Diagram 4: A crossing with strands } a, b. A blue dot is on strand } a \text{ below the crossing.} \end{array} & \mapsto & \begin{array}{c} \text{Diagram 5: A crossing with strands } a, b. \end{array} \\ \begin{array}{c} \text{Diagram 6: A crossing with strands } a, b, c. A red dot is on strand } a \text{ below the crossing.} \end{array} & \mapsto & 0. \end{array}$$

**Theorem 1.** *The chain maps  $\pi_3$  and  $\pi_2$  satisfy the following.*

$$(3) \quad h_2 \circ \pi_3 = \pi_3 \circ h_3,$$

$$(4) \quad h_1 \circ \pi_2 = \pi_2 \circ h_2.$$

*Proof.*  $\delta_{s,t} \circ \pi_J = \pi_J \circ \delta_{s,t}$  and  $h_{J-1} \circ \pi_J = \pi_J \circ h_J$  ( $J = 2, 3$ ) are proved by direct computation for every generator of  $\mathcal{C}_J$ .  $\square$

$$\text{Let } \mathcal{C}'_3 = \mathcal{C} \left( \begin{array}{ccc} \begin{array}{c} \text{Diagram 7: Crossing with strands } p, q, r. Blue dots on } r \text{ and } q. \end{array} \otimes [xa] + \begin{array}{c} \text{Diagram 8: Crossing with strands } p, q, \tilde{r}. Blue dots on } \tilde{r} \text{ and } q. \end{array} \otimes [xb], \begin{array}{c} \text{Diagram 9: Crossing with strands } p, q. \end{array} \otimes [x] \right) \text{ and} \\ \mathcal{C}'_2 = \mathcal{C} \left( \begin{array}{ccc} \begin{array}{c} \text{Diagram 10: Crossing with strands } p, q. Red dots on } p \text{ and } q. \end{array} \otimes [xa] + \begin{array}{c} \text{Diagram 11: Crossing with strands } p, q. Blue dots on } p \text{ and } q. \end{array} \otimes [xb] \right). \mathcal{C}'_J \text{ is a subcomplex of } \mathcal{C}_J \text{ (} J = 2, 3 \text{).}$$

We define  $\pi'_3 : \mathcal{C}'_3 \rightarrow \mathcal{C}_2$  by (1).

**Theorem 2.**

$$(5) \quad \rho_2 \circ \pi_3 = \pi'_3 \circ \rho_3.$$

*Proof.*  $\delta_{s,t} \circ \pi'_3 = \pi'_3 \circ \delta_{s,t}$  and  $\rho_2 \circ \pi_3 = \pi'_3 \circ \rho_3$  are proved by direct computation for every generator of  $\mathcal{C}_3$ .  $\square$

### 3. A SIMILAR MAP $\tilde{\pi}_2$ TO $\pi_2$ .

In this section we introduce a map  $\tilde{\pi}_2$ . It is not chain maps, but it has similar property  $h_{1'} \circ \tilde{\pi}_2 = \tilde{\pi}_2 \circ h_{1'}$  (Theorem 3).

The map  $\tilde{\pi}_2 : \mathcal{C}_2 \rightarrow \mathcal{C}_{1'}$  is defined by

$$\begin{aligned}
 (6) \quad & \begin{array}{c} \text{Diagram 1} \end{array} \mapsto \begin{array}{c} \text{Diagram 2} \\ \text{Diagram 3} \end{array} \\
 (7) \quad & \begin{array}{c} \text{Diagram 4} \end{array} \mapsto 0.
 \end{aligned}$$

The diagrams are as follows: (6) shows a crossing labeled 'a' with a red dot labeled 'b' on the upper strand, mapping to two diagrams: the first is a crossing labeled 'a' with a red dot labeled 'b' on the upper strand, and the second is a crossing labeled 'a' with a red dot labeled 'b' on the lower strand. (7) shows a crossing labeled 'a' with a blue dot labeled 'b' on the upper strand, mapping to 0.

### Theorem 3.

$$(8) \quad h_{1'} \circ \tilde{\pi}_2 = \tilde{\pi}_2 \circ h_{1'}.$$

*Proof.*  $h_{1'} \circ \tilde{\pi}_2 = \tilde{\pi}_2 \circ h_{1'}$  is proved by direct computation for every generator of  $\mathcal{C}_2$ .  $\square$

### 4. RIGHT TWISTED FIRST REIDEMEISTER INVARIANCE FOR THE GENERAL DIFFERENTIAL $\delta_{s,t}$ .

In this section we will show that the right twisted first Reidemeister invariance of Khovanov homology because the proof of this case is missing in [2].

The right twisted first Reidemeister move is  $D' = a \smile \sim \smile = D$ , we consider the composition

$$(9) \quad \mathcal{C}(D') = \mathcal{C} \oplus \mathcal{C}_{\text{contr}} \xrightarrow{\rho_1} \mathcal{C} \xrightarrow{\text{isom}} \mathcal{C}(D)$$

where  $a$  is a crossing and  $\mathcal{C}$ ,  $\mathcal{C}_{\text{contr}}$ ,  $\rho_1$  and the isomorphism are defined in the following formulas (10)–(13).

First,

$$\begin{aligned}
 (10) \quad \mathcal{C} &:= \mathcal{C} \left( p \smile \ominus \otimes [x] \right), \\
 \mathcal{C}_{\text{contr}} &:= \mathcal{C} \left( p : p \smile \smile \otimes [x], \smile : p \smile \otimes [xa] \right).
 \end{aligned}$$

Second, the retraction  $\rho_1 : \mathcal{C} \left( \smile \right) \rightarrow \mathcal{C} \left( p \smile \ominus \otimes [x] \right)$  is defined by the formulas

$$\begin{aligned}
(11) \quad & p \text{ (red dot) } \diagdown \bigcirc \otimes [x] \mapsto p \text{ (red dot) } \diagdown \bigcirc \otimes [x], \\
& p \text{ (red dot) } \diagup \bigcirc \otimes [x] \mapsto p \text{ (red dot) } \diagup \bigcirc \otimes [x] - p : p \text{ (red dot) } \diagup \bigcirc \otimes [x], \\
& \text{blue dot } \diagdown \bigcirc \otimes [xa] \mapsto 0.
\end{aligned}$$

We can verify that  $\delta_{s,t} \circ \rho_1 = \rho_1 \circ \delta_{s,t}$ . Then  $\rho_1$  is a chain map.

Third, the isomorphism

$$(12) \quad \mathcal{C} \left( p \text{ (red dot) } \diagdown \bigcirc \otimes [x] \right) \rightarrow \mathcal{C} \left( \bigcirc \otimes [x] \right)$$

is defined by the formulas

$$(13) \quad p \text{ (red dot) } \diagdown \bigcirc \otimes [x] \mapsto \bigcirc \otimes [x].$$

The homotopy connecting  $\text{id} \circ \rho_1$  to the identity  $: \mathcal{C} \left( \bigcirc \otimes [x] \right) \rightarrow \mathcal{C} \left( \bigcirc \otimes [x] \right)$  such that  $\delta_{s,t} \circ h_1 + h_1 \circ \delta_{s,t} = \text{id} - \text{id} \circ \rho_1$ , is defined by the formulas:

$$(14) \quad p \text{ (red dot) } \diagdown \bigcirc \otimes [xa] \mapsto \text{blue dot } \diagdown \bigcirc \otimes [x], \text{ otherwise } \mapsto 0.$$

**Remark 1.** The explicit formula (14) of the homotopy map  $h_1$  in the case ( $s = t = 0$ ) of the original Khovanov homology is given by Oleg Viro [3, Subsection 5.5].

We can verify  $\delta_{s,t} \circ h_1 + h_1 \circ \delta_{s,t} = \text{id} - \text{id} \circ \rho_1$  by a direct computation as follows.

$$\begin{aligned}
(15) \quad & (h_1 \circ \delta_{s,t} + \delta_{s,t} \circ h_1) \left( \text{blue dot } \diagdown \bigcirc \otimes [x] \right) = h_1 \left( p : p \text{ (red dot) } \diagup \bigcirc \otimes [xa] \right) \\
& = \text{blue dot } \diagdown \bigcirc \otimes [x] \\
& = (\text{id} - \rho_1) \left( \text{blue dot } \diagdown \bigcirc \otimes [x] \right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
(16) \quad & (h_1 \circ \delta_{s,t} + \delta_{s,t} \circ h_1) \left( p \text{ (red dot) } \diagup \bigcirc \otimes [x] \right) = p : p \text{ (red dot) } \diagup \bigcirc \otimes [x] \\
& = (\text{id} - \rho_1) \left( p \text{ (red dot) } \diagup \bigcirc \otimes [x] \right),
\end{aligned}$$

$$(17) \quad (h_1 \circ \delta_{s,t} + \delta_{s,t} \circ h_1) \left( p \text{ (red dot) } \diagdown \bigcirc \otimes [x] \right) = 0 = (\text{id} - \rho_1) \left( p \text{ (red dot) } \diagdown \bigcirc \otimes [x] \right).$$

## APPENDIX A. CHAIN HOMOTOPY MAPS.

The homotopy connecting  $\text{id} \circ \rho_{1'}$  to the identity  $h_{1'} : \mathcal{C} \left( \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right) \rightarrow \mathcal{C} \left( \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right)$  such that  $\delta_{s,t} \circ h_{1'} + h_{1'} \circ \delta_{s,t} = \text{id} - \text{id} \circ \rho_{1'}$ , is defined by the formulas:

$$(18) \quad \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \otimes [xa] \mapsto \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \otimes [x], \text{ otherwise } \mapsto 0.$$

The homotopy connecting  $\text{id} \circ \rho_2$  to the identity  $h_2 : \mathcal{C} \left( \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right) \rightarrow \mathcal{C} \left( \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right)$  such that  $\delta_{s,t} \circ h_2 + h_2 \circ \delta_{s,t} = \text{id} - \text{id} \circ \rho_2$ , is defined by the formulas:

$$(19) \quad \begin{array}{c} p \\ \diagup \diagdown \\ \diagdown \diagup \\ q \end{array} \otimes [xab] \mapsto - \begin{array}{c} p \\ \diagup \diagdown \\ \diagdown \diagup \\ q \end{array} \otimes [xb], \quad \begin{array}{c} p \\ \diagup \diagdown \\ \diagdown \diagup \\ q \end{array} \otimes [xb] \mapsto \begin{array}{c} p \\ \diagup \diagdown \\ \diagdown \diagup \\ q \end{array} \otimes [x],$$

otherwise  $\mapsto 0$ .

The homotopy connecting  $\text{id} \circ \rho_3$  to the identity, that is, a map  $h_3 : \mathcal{C} \left( \begin{array}{c} c \diagdown b \\ a \diagup \end{array} \right) \rightarrow \mathcal{C} \left( \begin{array}{c} c \diagdown b \\ a \diagup \end{array} \right)$  such that  $\delta_{s,t} \circ h_3 + h_3 \circ \delta_{s,t} = \text{id} - \text{id} \circ \rho_3$ , is defined by the formulas:

$$(20) \quad \begin{array}{c} r \diagdown p \\ \diagdown \diagup \\ q \end{array} \otimes [xab] \mapsto - \begin{array}{c} r \diagdown p \\ \diagdown \diagup \\ q \end{array} \otimes [xb], \quad \begin{array}{c} r \diagdown p \\ \diagdown \diagup \\ q \end{array} \otimes [xb] \mapsto \begin{array}{c} r \diagdown p \\ \diagdown \diagup \\ q \end{array} \otimes [x],$$

otherwise  $\mapsto 0$

## APPENDIX B. RETRACTIONS.

The retraction  $\rho_{1'} : \mathcal{C} \left( \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right) \rightarrow \mathcal{C} \left( \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right) \otimes [x] - \text{m}(p : +) \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \otimes [x] \right)$  is defined by the formulas

$$\begin{aligned}
(21) \quad & p \text{ (blue dot) } \text{---} \bigoplus \otimes [x] \mapsto p \text{ (blue dot) } \text{---} \bigoplus \otimes [x] - m(p : +) \text{---} \bigotimes \otimes [x], \\
& p \text{ (blue dot) } \text{---} \bigotimes \otimes [x], \quad \text{---} \bigoplus \otimes [xa] \mapsto 0.
\end{aligned}$$

The retraction  $\rho_2 : \mathcal{C} \left( \begin{array}{c} a \\ b \end{array} \right) \rightarrow \mathcal{C} \left( \begin{array}{c} p \text{ (blue dot) } \text{---} q \\ q \text{ (blue dot) } \text{---} p \end{array} \right) \otimes [xa] + \begin{array}{c} p : q \\ \text{---} \\ q : p \end{array} \otimes [xb] \right)$  is defined by the formulas

$$\begin{aligned}
(22) \quad & \begin{array}{c} p \text{ (blue dot) } \text{---} q \\ q \text{ (blue dot) } \text{---} p \end{array} \otimes [xa] \mapsto \begin{array}{c} p \text{ (blue dot) } \text{---} q \\ q \text{ (blue dot) } \text{---} p \end{array} \otimes [xa] + \begin{array}{c} p : q \\ \text{---} \\ q : p \end{array} \otimes [xb], \\
& \begin{array}{c} p \\ \text{---} \bigoplus \\ q \end{array} \otimes [xb] \mapsto - \left( \begin{array}{c} p : q \text{ (blue dot) } \text{---} q : p \end{array} \otimes [xa] + \begin{array}{c} (p : q) : (q : p) \\ \text{---} \\ (q : p) : (p : q) \end{array} \otimes [xb] \right), \\
& \text{otherwise} \mapsto 0.
\end{aligned}$$

The retraction  $\rho_3 : \mathcal{C} \left( \begin{array}{c} c \\ a \end{array} \right) \rightarrow \mathcal{C} \left( \begin{array}{c} r \text{ (blue dot) } \text{---} p \\ q \text{ (blue dot) } \text{---} p \end{array} \right) \otimes [xa] + \begin{array}{c} \tilde{r} \\ \text{---} \\ p : q \end{array} \otimes [xb],$

$\begin{array}{c} \text{---} \bigoplus \\ \text{---} \end{array} \otimes [x]$  is defined by the formulas

$$\begin{aligned}
(23) \quad & \begin{array}{c} r \text{ (blue dot) } \text{---} p \\ q \text{ (blue dot) } \text{---} p \end{array} \otimes [xa] \mapsto \begin{array}{c} r \text{ (blue dot) } \text{---} p \\ q \text{ (blue dot) } \text{---} p \end{array} \otimes [xa] + \begin{array}{c} \tilde{r} \\ \text{---} \\ p : q \end{array} \otimes [xb], \\
& \begin{array}{c} \text{---} \bigoplus \\ \text{---} \end{array} \otimes [x] \mapsto \begin{array}{c} \text{---} \bigoplus \\ \text{---} \end{array} \otimes [x],
\end{aligned}$$

$$\begin{array}{c}
\begin{array}{c} \text{Diagram 1: Crossing with blue dots on top and bottom strands, red dots on the crossing. Labeled } r \text{ and } q. \end{array} \otimes [xb] \mapsto - \begin{array}{c} \text{Diagram 2: Crossing with blue dots on top and bottom strands, red dots on the crossing. Labeled } r \text{ and } p:q. \end{array} \otimes [xa] - \begin{array}{c} \text{Diagram 3: Crossing with blue dots on top and bottom strands, red dots on the crossing. Labeled } \tilde{r} \text{ and } (p:q):(q:p). \end{array} \otimes [xb] - \begin{array}{c} \text{Diagram 4: Crossing with blue dots on top and bottom strands, red dots on the crossing. Labeled } r:p \text{ and } q. \end{array} \otimes [xc], \\
\begin{array}{c} \text{Diagram 5: Crossing with blue dots on top and bottom strands, red dots on the crossing. Labeled } r \text{ and } q. \end{array} \otimes [xab] \mapsto \begin{array}{c} \text{Diagram 6: Crossing with blue dots on top and bottom strands, red dots on the crossing. Labeled } r \text{ and } q. \end{array} \otimes [abc],
\end{array}$$

otherwise  $\mapsto 0$ .

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DEPARTMENT OF PURE AND APPLIED MATHEMATICS WASEDA UNIVERSITY. TOKYO 169-8555, JAPAN.  
*E-mail address:* noboru@moegi.waseda.jp